Differentiation (?)!

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Let \mathbb{D} be the unit disc in the complex plane \mathbb{C} . This is just the inside of the unit circle \mathbb{T} . We have functions defined on it

 $-\varphi_{\alpha,a}(z) = \alpha \frac{z-a}{1-\bar{a}z}$, where *a* is in \mathbb{D} and α is in \mathbb{T} . Each of these is the ratio of two polynomials of degree one. One can't ask for anything simpler!

If we go forward from \mathbb{D} with one these functions, then we land in \mathbb{D} again. Moreover, we always have a function of the form $\varphi_{\beta,b}$ with which we can return back to the first copy of the \mathbb{D} undoing the effect of $\varphi_{\alpha,a}$.

We say that each of the $\varphi_{\alpha,a}$ admits an inverse, namely the function $\varphi_{\beta,b}$. The set of these functions, namely, $\{\varphi_{\alpha,a} : \alpha \in \mathbb{T}, a \in \mathbb{D}\}$ forms a group G under composition of functions and is called the Möbius group



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Consider the space $L^2_a(\mathbb{D})$ of all holomorphic functions on \mathbb{D} which are square integrable with respect to the area measure. This consists of the functions (these are polynomials that refuse to stop):

 $\{f: f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots\}$

with $|a_0|^2 + |a_1|^2 + 2|a_2|^2 + \dots + (n+1)|a_n|^2 + \dots < \infty$. The space $L^2_a(\mathbb{D})$ of functions defined on \mathbb{D} is a Hilbert space. Let $\mathcal{A}(\mathbb{D})$ be the set of holomorphic functions (again, polynomials that refuse to stop) which are continuous on the union of the two sets \mathbb{D} and \mathbb{T} , it is an algebra. We have described three mathematical objects, namely the Möbius group G, the Hilbert space $L^2_a(\mathbb{D})$, and the algebra $\mathcal{A}(\mathbb{D})$. Consider the space $L^2_a(\mathbb{D})$ of all holomorphic functions on \mathbb{D} which are square integrable with respect to the area measure. This consists of the functions (these are polynomials that refuse to stop):

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while the action ϱ of the algebra $\mathcal{A}(\mathbb{D})$ is obtained by a mere multiplication –

 $\big(\varrho(f)h\big)(z)=f(z)h(z),\,h\in L^2_a(\mathbb{D})$

What is more, U is a (actually, in general, projective) group homomorphism and ρ is an algebra homomorphism. These satisfy the imprimitivity relation (a form of Weyl commutation relation):

 $arrho(arphi \cdot f) = U(arphi)^* \varrho(f) U(arphi), \ f \in \mathcal{A}(\mathbb{D}), \ \varphi \in G$ where $\varrho(arphi \cdot f)(z) = \varphi(z) f(z).$



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Let \mathcal{H} be a space of functions, say, on the unit disc or the unit circle. Suppose that the homomorphism $\varrho : \mathcal{A}(\mathbb{D}) \to \mathcal{L}(\mathcal{H})$ defined by the rule $\varrho(f) = M_f, f \in \mathcal{A}(\mathbb{D})$ is bounded. Let $U: G \to \mathcal{L}(\mathcal{H})$ be of the form $U(\varphi) = M_{J_{\varphi}}R_{\varphi}$, where $M_{J_{\varphi}}$ is the multiplication by J_{φ} and R_{φ} is the composition by φ . The map U is a homomorphism if and only if the multiplier identity

 $J_{\varphi\psi}(z) = J_{\varphi}(\psi(z))J_{\psi}(z), \, \varphi, \psi \in G$

is valid for the function $J: G \times \mathbb{D} \to \mathbb{C}$. In this case U is said to be a multiplier representation.

If there is a multiplier representation, say U, of the group G on the Hilbert space \mathcal{H} , then the imprimitivity relationship

 $\left(M_{J_{\varphi}}R_{\varphi}\right)^* \varrho(\varphi \cdot f)(M_{J_{\varphi}}R_{\varphi}) = \varrho(f), \varphi \in G f \in \mathcal{A}(\mathbb{D}).$



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Let us also emphasize that the map $U: G \to \mathcal{O}(\mathbb{D})$ defined by the rule $(U(\varphi)f)(z) = J_{\varphi}(z)f(\varphi(z))$ is a homomorphism only if J satisfies the multiplier identity.

Clearly, any power of the derivative $J_{\varphi}^{(\lambda)} := (\varphi')^{\lambda}, \lambda > 0$ will continue to obey the multiplier identity.

Surprisingly, these are all the possible complex valued multipliers for the Möbius group.

Given the multiplier $J^{(\lambda)}$, it is easy to find a Hilbert space $\mathcal{H}^{(\lambda)}$ such that U and ϱ , defined as before, acts on it satisfying the imprimitivity condition.

How do we construct multipliers taking values, say, in $n \times n$ matrices?





Recall that the multiplier identity for $J: G \times \mathbb{D} \to \mathbb{C}$, where J_{φ} is the derivative φ' , is the familiar chain rule. Let us also emphasize that the map $U: G \to \mathcal{O}(\mathbb{D})$ defined by the rule $(U(\varphi)f)(z) = J_{\varphi}(z)f(\varphi(z))$ is a homomorphism only if J satisfies the multiplier identity.

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How do we construct multipliers taking values, say, in $n \times n$ matrices?



Assume that we have found a derivation $\mathfrak{d}: G \times \mathbb{D} \to \mathbb{C}^{n \times n}$ satisfying the multiplier identity, that is,

$\mathfrak{d}(gh,z)=\mathfrak{d}(g,h(z))\mathfrak{d}(h,z),\,g,h\in G,\,z\in\mathbb{D}.$

For $0 < \lambda \in \mathbb{R}$, define the map $\Gamma : G \to \mathcal{E}(\mathcal{O}(\mathbb{D}, \mathbb{C}^n))$ by the rule

 $\left(\Gamma(g^{-1})f\right)(z)=J^{(\lambda)}(g,z)\mathfrak{d}(g,z)f(g(z)),\ f\in \mathbb{O}(\mathbb{D},\mathbb{C}^n),\ g\in G.$

Not only Γ is a homomorphism but any homomorphism must be of this form. It would be therefore desirable to find all the possible derivations \mathfrak{d} .



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Let D(g, z) be the diagonal matrix whose (ℓ, ℓ) entry is $J^{(-m+j)}(g, z)I_{d_j}, \quad d_0 + \cdots + d_j < \ell \leq d_{j+1}, \quad d_0 + \cdots + d_m = n.$ Also, the ratio

$$-\frac{1}{2}\frac{g''(z)}{(g'(z))^{\frac{3}{2}}}, \ g \in G, \ z \in \mathbb{D}$$

is independent of z, which we denote by b_g . Let $Y_i : \mathbb{C}^{d_j} \to \mathbb{C}^{d_{j+1}}$ be a set of m linear transformations and Y be the corresponding shift operator on \mathbb{C}^n .



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The proof

The multiplier identity holds for the derivation \mathfrak{d} defined by the rule:

 $(g,z) \mapsto J^{(\lambda)}(g,z)D(g,z)^{\frac{1}{2}} \exp(-b_g Y)D(g,z)^{\frac{1}{2}}, g \in G, z \in \mathbb{D}.$

Proof: It is easy to verify that $D(g_1g_2,z) = D(g_1,g_2(z))D(g_2,z)$ using the chain rule. Now,

 $\mathfrak{d}(g_1g_2, z) = D(g_1g_2, z)^{\frac{1}{2}} \exp(-b_{g_1g_2}Y) D(g_1g_2, z)^{\frac{1}{2}}.$



proof contd.

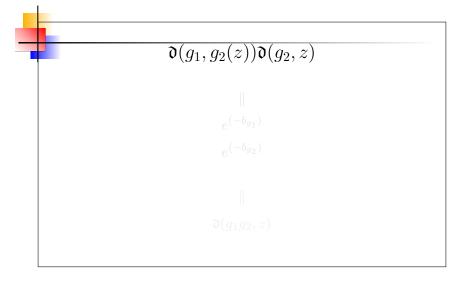
However, we have

$$\begin{aligned} -b_{g_{1}g_{2}} &= \frac{1}{2} \frac{(g_{1}g_{2})''(z)}{((g_{1}g_{2})'(z))^{3/2}} \\ &= \frac{1}{2} \frac{(g_{1}'(g_{2}(z))g_{2}'(z))'}{(g_{1}'(g_{2}(z))g_{2}'(z))^{3/2}} \\ &= \frac{1}{2} \frac{g_{1}''(g_{2}(z))(g_{2}'(z))^{2} + g_{1}'(g_{2}(z))g_{2}''(z)}{(g_{1}'(g_{2}(z))g_{2}'(z))^{3/2}} \\ &= \frac{1}{2} \{ \frac{g_{1}''(g_{2}(z))}{(g_{1}'(g_{2}(z)))^{3/2}}g_{2}'(z)^{1/2} + \frac{g_{2}''(z)}{(g_{2}'(z))^{3/2}}(g_{1}'(g_{2}(z)))^{-1/2} \} \\ &= -b_{g_{1}}(g_{2}'(z))^{1/2} - b_{g_{2}}(g_{1}'(g_{2}(z)))^{-1/2}. \end{aligned}$$

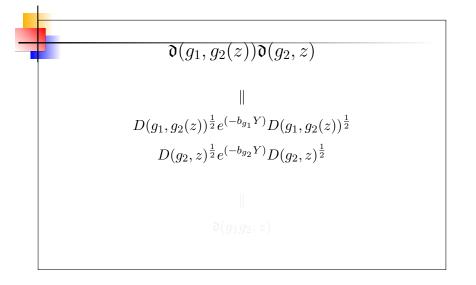




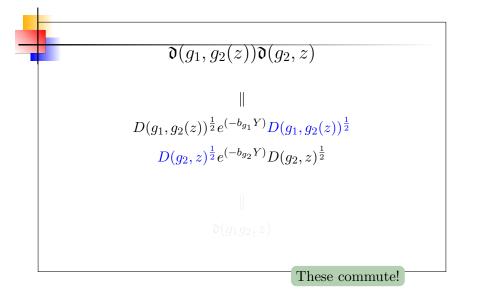




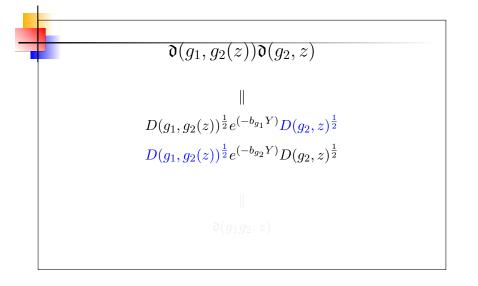




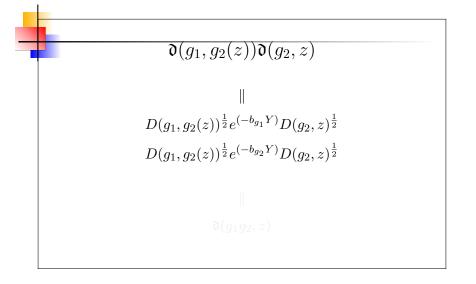




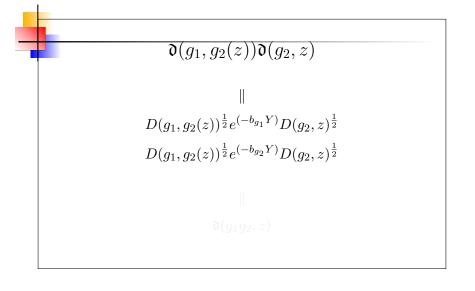






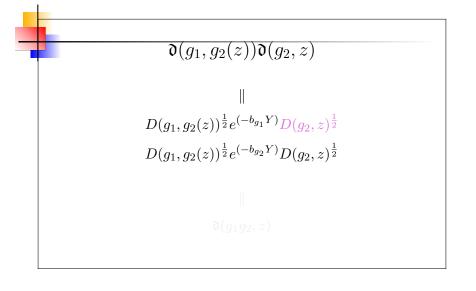






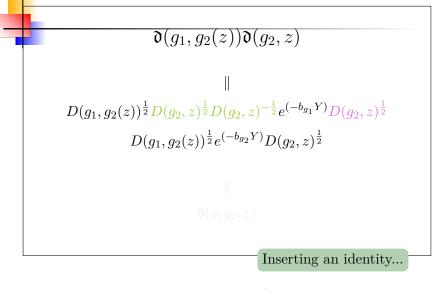
$$A^{-1}e^X A = e^{(A^{-1}XA)}$$





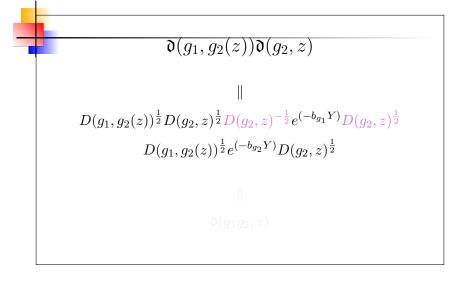
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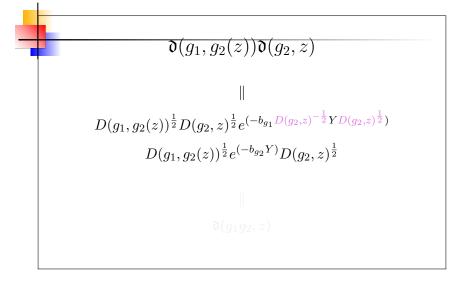
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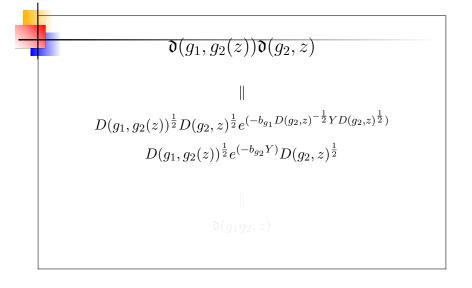
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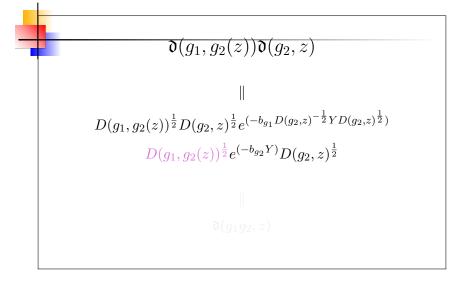
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$$\begin{split} \mathfrak{d}(g_1,g_2(z))\mathfrak{d}(g_2,z) \\ \| \\ D(g_1,g_2(z))^{\frac{1}{2}}D(g_2,z)^{\frac{1}{2}}e^{(-b_{g_1}D(g_2,z)^{-\frac{1}{2}}YD(g_2,z)^{\frac{1}{2}})} \\ D(g_1,g_2(z))^{\frac{1}{2}}e^{(-b_{g_2}Y)}D(g_1,g_2(z))^{-\frac{1}{2}}D(g_1,g_2(z))^{\frac{1}{2}}D(g_2,z)^{\frac{1}{2}} \\ \| \\ \mathfrak{d}(g_1g_2,z) \\ \end{split}$$

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 \parallel
 $D(g_1, g_2(z))^{rac{1}{2}}D(g_2, z)^{rac{1}{2}}e^{(-b_{g_1}D(g_2, z)^{-rac{1}{2}}YD(g_2, z)^{rac{1}{2}})}$
 $D(g_1, g_2(z))^{rac{1}{2}}e^{(-b_{g_2}Y)}D(g_1, g_2(z))^{-rac{1}{2}}D(g_1, g_2(z))^{rac{1}{2}}D(g_2, z)^{rac{1}{2}}$
 \parallel
 $\mathfrak{d}(g_1g_2, z)$

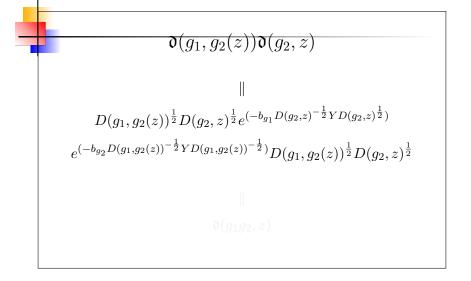
$$A^{-1}e^X A = e^{(A^{-1}XA)}$$



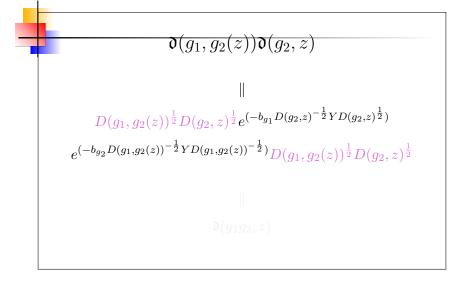
$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z) \\ \parallel \\ D(g_1, g_2(z))^{\frac{1}{2}}D(g_2, z)^{\frac{1}{2}}e^{(-b_{g_1}D(g_2, z)^{-\frac{1}{2}}YD(g_2, z)^{\frac{1}{2}})} \\ e^{(-b_{g_2}D(g_1, g_2(z))^{-\frac{1}{2}}YD(g_1, g_2(z))^{-\frac{1}{2}})}D(g_1, g_2(z))^{\frac{1}{2}}D(g_2, z)^{\frac{1}{2}} \\ \parallel \\ \mathfrak{d}(g_1g_2, z)$$

$$A^{-1}e^X A = e^{(A^{-1}XA)}$$

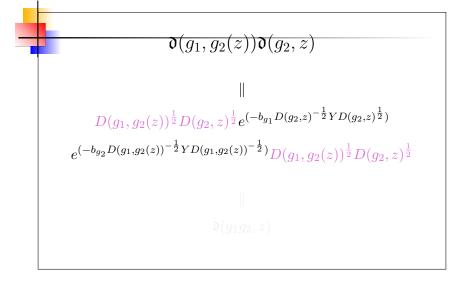












$$D(g_2, z)^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} = D(g_1g_2, z)^{\frac{1}{2}}$$



$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

 \parallel
 $D(g_1g_2, z)^{rac{1}{2}}e^{(-b_{g_1}D(g_2, z)^{-rac{1}{2}}YD(g_2, z)^{rac{1}{2}})}$
 $e^{(-b_{g_2}D(g_1, g_2(z))^{-rac{1}{2}}YD(g_1, g_2(z))^{-rac{1}{2}})}D(g_1g_2, z)^{rac{1}{2}}$
 \parallel
 $\mathfrak{d}(g_1g_2, z)$

$$D(g_2, z)^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} = D(g_1g_2, z)^{\frac{1}{2}}$$



$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

 \parallel
 $D(g_1g_2, z)^{\frac{1}{2}}e^{(-b_{g_1}D(g_2, z)^{-\frac{1}{2}}YD(g_2, z)^{\frac{1}{2}})}$
 $e^{(-b_{g_2}D(g_1, g_2(z))^{-\frac{1}{2}}YD(g_1, g_2(z))^{-\frac{1}{2}})}D(g_1g_2, z)^{\frac{1}{2}}$
 \parallel
 $\mathfrak{d}(g_1g_2, z)$

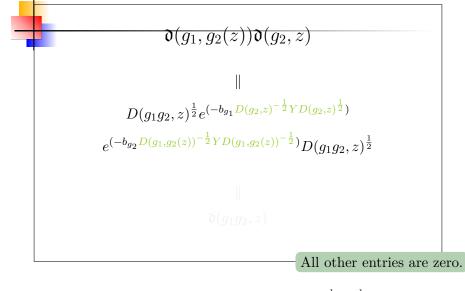
$$D(g_2, z)^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} = D(g_1g_2, z)^{\frac{1}{2}}$$



$$\begin{split} \mathfrak{d}(g_1,g_2(z))\mathfrak{d}(g_2,z) \\ \| \\ D(g_1g_2,z)^{\frac{1}{2}}e^{(-b_{g_1}D(g_2,z)^{-\frac{1}{2}}YD(g_2,z)^{\frac{1}{2}})} \\ e^{(-b_{g_2}D(g_1,g_2(z))^{-\frac{1}{2}}YD(g_1,g_2(z))^{-\frac{1}{2}})}D(g_1g_2,z)^{\frac{1}{2}} \\ \| \\ \mathfrak{d}(g_1g_2,z) \end{split}$$

$$(D(g_2, z)^{-1/2} Y D(g_2, z)^{1/2})_{i(i+1)} = d_i^{-\frac{1}{2}} Y_i d_{(i+1)}^{\frac{1}{2}}$$





$$(D(g_2, z)^{-1/2} Y D(g_2, z)^{1/2})_{i(i+1)} = d_i^{-\frac{1}{2}} Y_i d_{(i+1)}^{\frac{1}{2}}$$



$$\begin{split} \mathfrak{d}(g_1,g_2(z))\mathfrak{d}(g_2,z) \\ \| \\ D(g_1g_2,z)^{\frac{1}{2}}e^{(-bg_1D(g_2,z)^{-\frac{1}{2}}YD(g_2,z)^{\frac{1}{2}})} \\ e^{(-bg_2D(g_1,g_2(z))^{-\frac{1}{2}}YD(g_1,g_2(z))^{-\frac{1}{2}})}D(g_1g_2,z)^{\frac{1}{2}} \\ \| \\ \mathfrak{d}(g_1g_2,z) \end{split}$$

$$(D(g_2, z)^{-1/2} Y D(g_2, z)^{1/2})_{i(i+1)} = d_i^{-\frac{1}{2}} Y_i d_{(i+1)}^{\frac{1}{2}}$$



$$\begin{split} \mathfrak{d}(g_1,g_2(z))\mathfrak{d}(g_2,z) \\ \| \\ D(g_1g_2,z)^{\frac{1}{2}}e^{(-b_{g_1}D(g_2,z)^{-\frac{1}{2}}YD(g_2,z)^{\frac{1}{2}})} \\ e^{(-b_{g_2}D(g_1,g_2(z))^{-\frac{1}{2}}YD(g_1,g_2(z))^{-\frac{1}{2}})}D(g_1g_2,z)^{\frac{1}{2}} \\ \| \\ \mathfrak{d}(g_1g_2,z) \end{split}$$

$$(D(g_2,z)^{-1/2}YD(g_2,z)^{1/2})_{i(i+1)} = d_i^{-\frac{1}{2}} d_{(i+1)}^{\frac{1}{2}} Y_i$$



$$\begin{split} \mathfrak{d}(g_1,g_2(z))\mathfrak{d}(g_2,z) \\ \| \\ D(g_1g_2,z)^{\frac{1}{2}}e^{(-b_{g_1}D(g_2,z)^{-\frac{1}{2}}YD(g_2,z)^{\frac{1}{2}})} \\ e^{(-b_{g_2}D(g_1,g_2(z))^{-\frac{1}{2}}YD(g_1,g_2(z))^{-\frac{1}{2}})}D(g_1g_2,z)^{\frac{1}{2}} \\ \| \\ \mathfrak{d}(g_1g_2,z) \end{split}$$

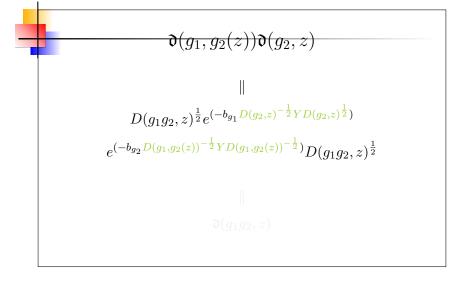
$$(D(g_2,z)^{-1/2}YD(g_2,z)^{1/2})_{i(i+1)} = (d_{i+1}/d_i)^{\frac{1}{2}}Y_i$$



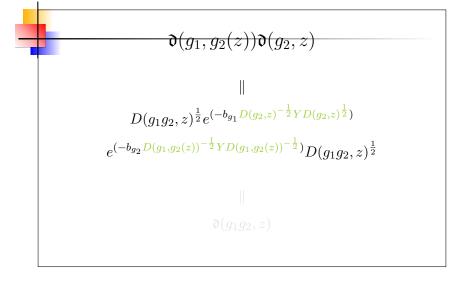
$$\begin{split} \mathfrak{d}(g_1,g_2(z))\mathfrak{d}(g_2,z) \\ \| \\ D(g_1g_2,z)^{\frac{1}{2}}e^{(-b_{g_1}D(g_2,z)^{-\frac{1}{2}}YD(g_2,z)^{\frac{1}{2}})} \\ e^{(-b_{g_2}D(g_1,g_2(z))^{-\frac{1}{2}}YD(g_1,g_2(z))^{-\frac{1}{2}})}D(g_1g_2,z)^{\frac{1}{2}} \\ \| \\ \mathfrak{d}(g_1g_2,z) \end{split}$$

$$(D(g_2, z)^{-1/2} Y D(g_2, z)^{1/2})_{i(i+1)} = (g'_2(z))^{\frac{1}{2}} Y_i$$

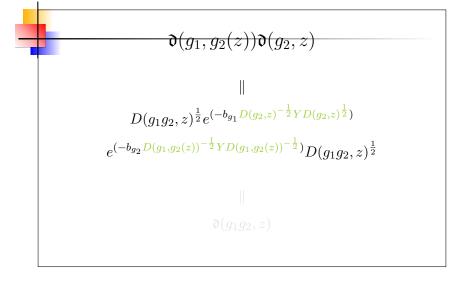




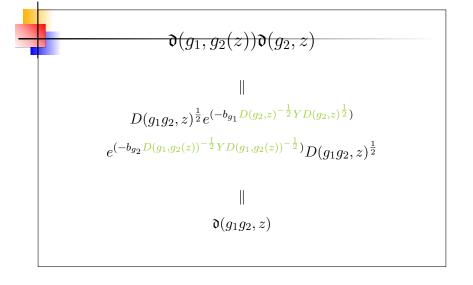
















Thank you!

